# Time-Optimal Turn to a Heading: An Analytic Solution

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## I. Introduction

THIS paper deals with the problem of the time-optimal turn of an aircraft to a given heading using a special simplified model. We solve this problem analytically. The result is an easy-to-apply feedback law. We use a quadratic drag law. The investigation is limited to a horizontal turn, neglecting the effects of the gravitational forces. Nevertheless it gives insight into the structure of the problem.

The problem of optimal turns of aircraft is widely discussed in the literature. The time-optimal turn to a heading is a very special and, when confined to the horizontal plane, a relatively simple problem. Much work has been done in this area, but it differs in modeling, approach, and results (see, for example, Refs. 1–4). Most approaches are numerical solution procedures. The aim of this paper is to develop a feedback strategy with analytical tools that is easily applied.

## II. Problem

An aircraft moves in a horizontal plane (flat earth, no wind) with initial speed $v_0$ and with initial heading angle $\chi_0$. It seeks to change its heading in the shortest possible time through the angle $\Delta$ to the final angle $\chi_f$. The initial time is $t_0 = 0$, and the unknown final time is $t_f$. While the aircraft is turning, the speed changes from $v_0$ up or down to $v_f$. We assume that the final speed is given, because otherwise the optimal control consists of the quickest possible turn without respect to any loss of speed.

Changes of speed and heading are given by the differential equations

$$\dot{v} = f_0(v) - Cu^2 \tag{1}$$

$$\dot{\chi} = u \tag{2}$$

The control function, which we want to choose in an optimal way, is $u$: $[0, t_f] \to R$. Physically $u$ is the turn rate of the aircraft. We installed in Eq. (1) a quadratic drag law. In this simple mathematical model $C$ has a constant positive value. $f_0 = (T - D_0)/m$ is the acceleration in straight line flight. We suppose that $f_0 > 0$, which means that the aircraft is able to accelerate in a straight line flight. The variable $m$ is the mass of the aircraft, $T$ is the constant thrust, $D_0 = ½\rho S v^2 c_{D_0}$ is the drag in straight line flight, which typically increases quadratically with the speed, $\rho$ is the air pressure, $S$ is the aerodynamically efficient surface, and $c_{D_0}$ is the drag coefficient. We assume all of these values to be constant.

This modeling differs from the usual one, for example, the one used in Ref. 3. A short explanation might be appropriate. The normal acceleration force of an aircraft flying a curved path in two-dimensional or three-dimensional space is $mvu$, where $u$ is the turn rate of the tangent vector of the trajectory. This force is equal to the normal component of the other forces involved. If we neglect all forces but the lift, we get

$$mvu = ½\,\rho S v^2 c_L$$

We solve for $c_L$ and insert it into

$$D = D_0 + ½\,\rho S v^2 K c_L^2$$

Substituting $D$ in $\dot{v} = (T - D)/m$ results in Eq. (1) with $C = (2mK/\rho S)$. Mainly we have neglected the normal component of the gravitational force. I found this necessary to obtain an analytic solution. The compromise may be acceptable if gravitation is small in comparison with the lift.

The problem is to find the optimal control $u$ as a function of time $t$ (open-loop control) or even better as a function of the state variables (feedback control), which turns the aircraft in the shortest possible time through the angle $\Delta = \chi_f - \chi_0 > 0$ (see Fig. 1).

## III. Necessary Optimality Conditions

The following derivation of the optimal control makes use of Pontryagin's maximum principle. Note that often the minimum formulation is used; see, for example, Ref. 5. We have the adjoint differential equations

$$\dot{\lambda}_v = -\lambda_v \frac{\partial f_0}{\partial v} \tag{3}$$

$$\dot{\lambda}_\chi = 0 \tag{4}$$

and the Hamiltonian function

$$H = \lambda_v[f_0(v) - Cu^2] + \lambda_\chi u \tag{5}$$

Necessary optimality conditions are (see Ref. 5, p. 71)

$$H(v, \chi, \lambda_v, \lambda_\chi, u)(t) = 1 \qquad \text{for all } t \in [0, t_f] \tag{6}$$

$$\frac{\partial H}{\partial u}(v, \chi, \lambda_v, \lambda_\chi, u)(t) = 0 \qquad \text{for all } t \in [0, t_f] \tag{7}$$

These equations make it possible to solve for the adjoint variables. This leads to the following.

*Lemma 1:*

There is a constant $u^* > 0$, such that the optimal control $u(t)$ satisfies

$$u^2 - 2u^*u + \frac{f_0(v)}{C} = 0 \qquad \text{for } t \in [0, t_f] \tag{8}$$

*Proof:*

We clearly assume $u(t) \neq 0$ in $(0, t_f)$, solve Eq. (7) for $\lambda_v$, and insert it in Eq. (6). The result is $\lambda_\chi(f_0 + Cu^2) = 2Cu$.

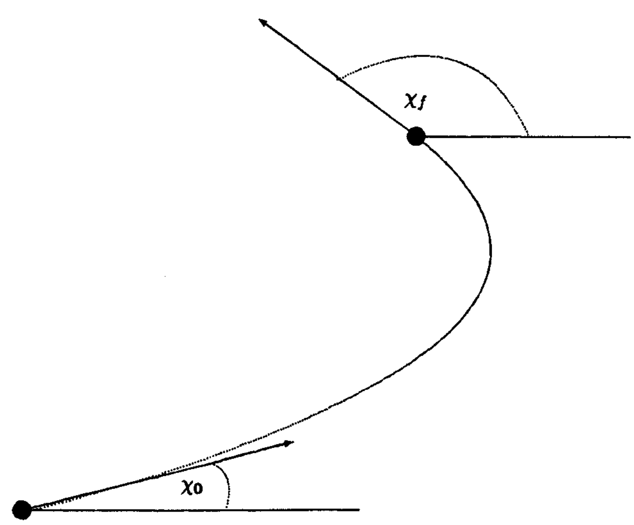


**Fig. 1 Turn of an aircraft in the horizontal plane.**

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Equation (4) shows that the function $\lambda_\chi$ is indeed a constant, with $\mathrm{sgn}(u) = \mathrm{sgn}(\lambda_\chi)$. With $u^* = 1/\lambda_\chi$ we get the assertion. □

Now we have obtained almost a feedback law. The variable $u$ still depends on the unknown constant $u^*$:

$$u(v) = u^* \pm \sqrt{u^{*2} - \frac{f_0}{C}} \tag{9}$$

The fact that $\chi$ is not involved in this formula becomes clear in realizing that the turn rate must be independent of the choice of the reference direction of the measurement of the angle $\chi$. The sign of $u^*$ apparently determines the sign of $u$, i.e., the turn direction. We assume $u^* > 0$, i.e., $\chi_f > \chi_0$ (turn to the left).

We still need a procedure to calculate the constant $u^*$ and a decision rule for the choice of the sign in Eq. (9). Now we necessarily need the initial conditions

$$v(0) = v_0, \qquad \chi(0) = \chi_0, \qquad v(t_f) = v_f, \qquad \chi(t_f) = \chi_f$$

With Eq. (9) the differential equation (1) changes to

$$\dot v = f_0 - C\left(u^* \pm \sqrt{u^{*2} - \frac{f_0}{C}}\right)^2 \tag{10}$$

As long as the radicand is positive, the right-hand side is Lipschitz continuous. Therefore the solution of the differential equation (10) (to given initial conditions) is unique. The choice of the positive sign in Eq. (10) results in a reduction of speed and the choice of the negative sign in an increase. This means that the sign is determined by the initial conditions. For $v_0 > v_f$ we choose the positive sign and for $v_0 < v_f$ the negative.

## IV. Calculation of the Constant $u^*$

The stationary speed $v_s(u^*)$, which belongs to the turn rate $u = u^*$, plays an important role. This is the speed at which it is possible to fly the turn rate without change of speed. The $v_s(u^*)$ is the positive solution of the quadratic equation $f_0 - Cu^{*2} = 0$. The control (9) guarantees that this speed is not crossed from below or above, that is, once we have reached a stationary turn with turn rate $u = u^*$, the control (9) does not change the speed, and a constant speed does not change the control. So we exclude the case that in $(0, t_f)$ we have a point with $\dot v = 0$. Now the function $t \to v(t)$ is invertible and

$$\mathrm{d}t = \frac{\mathrm{d}v}{f_0 - Cu^2}$$

We have

$$\Delta = \chi_f - \chi_0 = \int_0^{t_f} u(t)\,\mathrm{d}t = \int_{v_0}^{v_f} \frac{u}{f_0 - Cu^2}\,\mathrm{d}v$$

and with Eqs. (8) and (9)

$$\frac{u}{f_0 - Cu^2} = \frac{1}{2C}\frac{1}{u^* - u} = \pm\frac{1}{2C}\frac{1}{\sqrt{C_1 v^2 - C_2(u^*)}}$$

where $C_1 = (½\ \rho Scp_0/mC)$, $C_2(u^*) = u_0^2 - u^*2$, and $u_0 = \sqrt{(F/mC}$. We integrate

$$\int \frac{\mathrm{d}v}{\sqrt{C_1 v^2 - C_2}} = \frac{1}{\sqrt{C_1}}\, ln(2\sqrt{C_1}\sqrt{C_1 v^2 - C_2} + 2C_1 v) + K$$

It is always

$$0 < (u - u^*)^2 = C_1 v^2 - C_2(u^*) = u^{*2} - u_s^2(v)$$

where $u_s(v) = \sqrt{u_0^2 - C_1 v^2}$ denotes the stationary turn rate at the speed $v$.

The result of the integration is

$$2C\sqrt{C_1}\Delta = \left| ln\left[\frac{\sqrt{u^{*2} - u_s^2(v_f)} + \sqrt{C_1}v_f}{\sqrt{u^{*2} - u_s^2(v_0)} + \sqrt{C_1}v_0}\right]\right| =: \Phi(u^*) \tag{11}$$

The function $\Phi$ is defined for $u^* \ge u^*_{\min} = \max\{u_s(v_0), u_s(v_f)\} = u_s(\min\{v_0, v_f\})$ and *monotonically* decreasing, with the limit $\Phi(\infty) = 0$ (see Fig. 2, for example). Now we have the following.

*Lemma 2:*

For given initial and final speeds $v_0$ and $v_f$ and a given turn angle $\Delta \in [0, \Delta_{\max}]$ there exists exactly one solution $u^* \in [u^*_{\min}, \infty]$ of Eq. (11). The maximum allowable turn angle is $\Delta_{\max} = [\Phi\ (u^*_{\min})/2C\sqrt{C_1}]$. If $\Delta = \Delta_{\max}$, the solution is $u^* = u^*_{\min}$, and the turning ends in a stationary situation.

The calculation of $u^*$ also implies the calculation of the minimal turn time:

$$t_f = \int_{v_0}^{v_f} \frac{\mathrm{d}v}{f_0 - Cu^2} = \frac{1}{2C}\int_{v_0}^{v_f} \frac{\mathrm{d}v}{u(u^* - u)} = \frac{\Delta}{u^*} + \frac{1}{2Cu^*}\int_{v_0}^{v_f} \frac{1}{u}\,\mathrm{d}v$$

## V. Larger Turn Angles

If the desired turn angle $\Delta$ is larger than $\Delta_{\max}$, the previous calculations do not apply, and it is not clear what the optimal control looks like. The solution of the differential equation (10) is no longer unique, if the stationary turn speed $v_s(u^*_{\min})$ is reached. Possible controls to obtain larger turn angles $\Delta > \Delta_{\max}$, discussed in theory and practice, are, for example, the following:

1) After reaching the stationary turn rate, continue with constant speed $v_s(u^*_{\min})$ and constant turn rate until the desired turn angle is obtained. This control is automatically provided by Eq. (9).

2) Turn with an initial loss of speed from $v_u = v_s(u^*_{\min})$ to some $v_l < v_u$ and then fly with increasing speed from $v_l$ back to $v_u$. In both regions the time-optimal controls are determined from Eq. (9).

Which of the strategies needs less time is shown by the following calculation. A stationary turn with the turn rate $u = u^*_{\min}$ and the turn speed $v_s(u^*_{\min})$ through a turn angle $2\Delta$ requires the time

$$t_s = \frac{2\Delta}{u^*_{\min}}$$

A time-optimal nonstationary turn with first decreasing speed from $v_u = v_s(u^*_{\min})$ to a smaller speed $v_l$ through the angle $\Delta$, followed by an optimal turn with increasing speed from $v_l$ to $v_u$ through the same angle $\Delta$, requires the time

$$t_i = \frac{2\Delta}{u^*_{\min}} + \frac{1}{u^*_{\min}}\int_{v_l}^{v_u} \frac{1}{f_0}\sqrt{u^{*2} - \frac{f_0}{C}}\,\mathrm{d}v$$

Obviously the optimal alternative is to continue the turning in a stationary way. But since Eq. (10) fails to be Lipschitz continous along this path, the usual analysis fails.

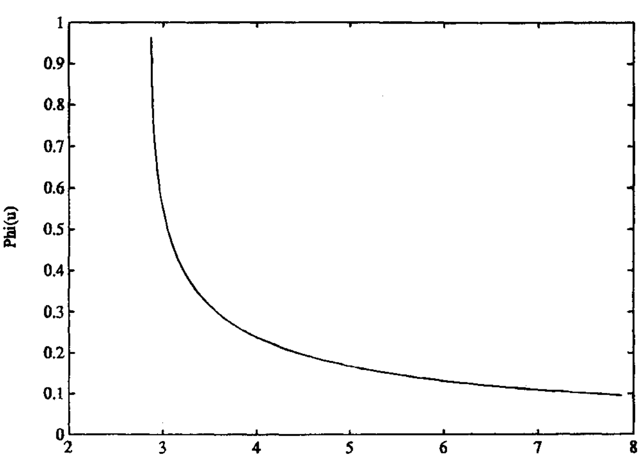


**Fig. 2 Graph of $\Phi$ with $v_0 = 300$ m/s, $v_f = 200$ m/s, $u^*_{\min} = 2.88$, $\Phi(u^*_{\min}) = 0.972$.**

## VI. Conclusion

This paper shows that there is a chance to find analytical or almost analytical feedback solutions for simplified models by searching for "integrals of motion." The feedback control obtained gives insight into the structure of the turning problem and may be used as a basis for the construction of more realistic controls.

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# Large Angle Slew Maneuvers with Autonomous Sun Vector Avoidance


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## Introduction

FREQUENT, large angle attitude slew maneuvers are required for certain space science missions for retargeting of payload instrumentation. For missions with sensitive payloads, such as cryogenically cooled infrared telescopes, the slew maneuver must be achieved without directing the payload along the Sun vector or at other infrared bright regions of the sky.[1] The planning of such constrained maneuvers can be costly in terms of increased ground segment work load and lack of system flexibility.

A methodology has been developed which allows large angle slew maneuvers to be achieved with autonomous avoidance of the Sun vector or other undesired orientations. The method is an extension of previous studies of large angle slews using the second method of Lyapunov.[2,3] The method is generalized by using artificial potential functions, where the local topology of the potential guides the satellite attitude during the slew maneuver. Undesired attitudes are then avoided by superimposing regions of high potential about these orientations.

The second method of Lyapunov allows expressions for the required control torques to be obtained analytically in closed form. Therefore, attitude control commands may be generated in real time, so that the method may be suitable for autonomous, onboard operations. Retargeting of payload instrumentation may then take place autonomously from a list of target objects to be viewed, thus reducing the ground segment loads and subsequent costs.

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## System Dynamics and Control

A general, rigid satellite will be considered rotating under the influence of body-fixed torquing devices only. The attitude motion may then be described by the standard Euler equations, viz.

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = T_1 \tag{1a}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = T_2 \tag{1b}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = T_3 \tag{1c}$$

where $\omega_i$, $I_i$, and $T_i$ ($i = 1, \ldots, 3$) are the satellite body rate, moment of inertia, and control torque about the $i$th principal axis. The body rates are related to the standard Euler angles through the kinematic relations

$$\dot{\theta}_i = \sum_{j=1}^{3} G_{ij}\omega_j,$$

$$\{G_{ij}\} = \begin{Bmatrix} 1 & \sin\theta_1\tan\theta_2 & \cos\theta_1\tan\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1\sec\theta_2 & \cos\theta_1\sec\theta_2 \end{Bmatrix} \tag{2}$$

A scalar potential function $V$ will be defined to be positive definite everywhere, except at the target point of the system state space where it will vanish. If an admissible control is then determined such that $\dot{V}$ is negative definite, Lyapunov's theorem guarantees that the satellite will slew to the target attitude from any initial orientation.

From Eq. (1), it may easily be verified that for open-loop, torque-free motion the total rotational kinetic energy of the system is conserved. Motivated by this, an initial potential function is chosen as

$$V = \frac{1}{2}\sum_{i=1}^{3} I_i\omega_i^2 + \frac{1}{2}\sum_{i=1}^{3}\lambda_i(\theta_i - \tilde{\theta}_i)^2 \tag{3}$$

where the second term represents an artificial potential energy possessed by the system relative to the target attitude ($\tilde{\theta}_1$, $\tilde{\theta}_2$, $\tilde{\theta}_3$). To obtain the required controls, a set of nonconservative torques are sought which render $\dot{V}$ negative definite. Differentiating the potential and substituting from Eq. (1), it is found that

$$\dot{V} = \sum_{i=1}^{3}\omega_i T_i + \sum_{i=1}^{3}\lambda_i\dot{\theta}_i(\theta_i - \tilde{\theta}_i) \tag{4}$$

where $\dot{\theta}_i$ ($i = 1, \ldots, 3$) are related to the body rates by Eq. (2). The choice of control is obviously non-unique; however, the simplest control is selected as

$$T_i = -\kappa\omega_i - \sum_{j=1}^{3}\{G_{ij}\}^T\lambda_j(\theta_j - \tilde{\theta}_j) \tag{5}$$

where the gain constants $\kappa$ and $\lambda_i$ ($i = 1, \ldots, 3$) will be chosen to shape the maneuver. Using these control torques, the rate of descent of the potential function then becomes

$$\dot{V} = -\kappa\sum_{i=1}^{3}\omega_i^2 \tag{6}$$

which is negative semidefinite. However, by substituting the